PHASE-PLANE MODELS IN KORELIA SOFTWARE

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That which is static and repetitive is boring.
That which is dynamic and random is confusing.
In between lies art.

John A. Locke (1632−1704),
British philosopher and medical researcher.

Abstract: In this work, the possibilities of the Korelia program for analysis of dynamic processes by the phase plane method are explored. The dynamics model is described using a specialized programming language. An algorithm for constructing nullclines of the process and for calculating equilibrium points is presented. The type of points is determined by the state matrix eigenvalues. All derived dependencies and parameters are displayed in graphical form. The approach is illustrated by a model of plasma renin activity after treatment of experimental animals with nicardipine. The special point is initial condition and it is of the stable focus type.
Key words: dynamics, modeling, stability, simulation, phase portrait, Korelia.

1. INTRODUCTION

The system stability test is related to solving the differential equation describing them. While for linear differential equations this is not a problem, it can be complicated in non-linear tasks [1]. An approach which can be used is to exclude the independent time parameter and to examine the behavior of the system in state space. Thus, the differential equation is studied using numerical or graphical methods without being solved. The result is a family of trajectories in a two-dimensional phase space, each corresponding to different initial conditions. These phase trajectories are

the tool for evaluating the behavior of dynamic systems - linear and nonlinear [2]. The phase plane method is a good tool when other approaches to finding analytical solutions of differential equations do not exist or are difficult to use. It is most often applied in systems described with differential equations to second order because the qualitative properties of the process are best represented graphically.

The purpose of this article is to propose an algorithm for studying of dynamic processes described by differential equations of the second order applying the phase plane method in the Korelia environment.

Korelia program is designed to process and analyze experimental data [3]. It offers a user-oriented interface [4], based on the presence of a language for describing the identification model [5] and a module for recognizing the studied process [6,7]. A two-way data transfer is possible by supporting file formats CSV, XLS, XML, Paradox, and SSP [8]. Experimental data can be identified with algebraic or differential models [9], analyzed and compared by a number of parameters [10]. Korelia program allows graphical visualization of data and results, and processing of the graphics image [11]. This is helpful for biology scientists, enabling them to apply a complex mathematical theory to study the dynamic behavior of the processes, using program with a friendly graphical interface.

2. PHASE PORTRAIT MODEL

Second-order ordinary differential equations have the form:

$$\frac{d^2 y(t)}{dt^2} + f\left(y(t), \frac{dy(t)}{dt} \right) = 0 \tag{1}$$

With the substitutions:

$$\left| \begin{array}{l} y_1 = y(t) \\ \\ y_2 = \dfrac{dy(t)}{dt} \end{array} \right. \tag{2}$$

It is reduced to a system of two autonomous differential equations from the first order:

$$\left| \begin{array}{l} \dfrac{dy_1}{dt} = f_1(y_1, y_2) \\ \\ \dfrac{dy_2}{dt} = f_2(y_1, y_2) \end{array} \right. \tag{3}$$

where: y_1, y_2 : defines the state space of the system, or phase plane of the system; f_1, f_2 : smooth nonlinear functions of the states.

Since the aim is to study the stability conditions, the system (3) is in equilibrium if the next condition is fulfilled:

$$\left|\begin{array}{l} f_1(y_1, y_2) = 0 \\ f_2(y_1, y_2) = 0 \end{array}\right. \tag{4}$$

Each of these equations defines graph in y_1Oy_2 coordinate system, known as nullclines of the process. The nullclines intersections $y^* = \left(y_1^*, y_2^*\right)$ determine the equilibrium points of the system. Thus the stability of systems is characterized by the nature of their equilibrium points. The ascertain of stability is done solving the algebraic system (4) instead of the differential equation (1). The system (4) can be solved analytically or by a numerical algorithm.

3. ANALYSIS OF SYSTEMS IN KORELIA ENVIRONMENT

Korelia is an interactive system capable of processing, identifying and simulating a wide range of processes described by algebraic and differential equations. The way to use its functions, as well as the approach to analyzing the results, is determined by the experience and qualification of the researcher. In the present work, a recommended algorithm for analysis of processes by the phase plane method is given.

3.1. Differential model of the studied system.

To analyze the data, it is necessary to identify them with a first or second order differential equation. Korelia system allows identification with models of random order [12]. If the model is already known, it is introduced with the language described in [5]. In the analysis of non-linear systems, a linearization with the Jacobi matrix is performed. According to Hartman-Grobman Theorem, near the equilibrium point *y**, the nonlinear system has the same quality as the linear system [13]. The system (3) has the matrix form:

$$\left\|\begin{array}{c} \dot{y}_1 \\ \dot{y}_2 \end{array}\right\| = \left\|\begin{array}{cc} a & b \\ c & d \end{array}\right\| . \left\|\begin{array}{c} y_1 \\ y_2 \end{array}\right\| \Rightarrow \dot{Y} = A.Y \tag{5}$$

In this case, the eigenvalues of the state matrix *A* are important, because they determine the type of system stability.

3.2. Determination of the phase plane.

According to the substitutions (2), the abscissa is represented by the independent variable *y(t)* and the ordinate axis is its first derivative $dy(t)/dt$. These coordinates form exemplar of class *KCartesian_System* [11].

3.3. Drawing the nullclines in the phase plane.

Two dynamic arrays *F1* and *F2* of class *KSpline* are created with the coordinates of calculated nullcline points. The number of intermediate points or sampling interval must be set at the discretion of the researcher. For the visualization of each of the two graphs, linear or spline interpolation is optionally used.

3.4. Identification of equilibrium points.

Equilibrium points are the intersections of nullclines. It is possible the points to be determined visually on the graphics. For precise calculation, a numeric algorithm is applied. The basic idea of the algorithm is to determine the vector between the two nullclines for the same abscissa. The change of sign of the vector means that the intersection is in the interval between the last two abscissa values. In the list of found intervals depending on the desired precision, the search for root is done by the bisection method [14]. The pseudo-code of the algorithm for constructing nullclines and calculating equilibrium points is in figure 1.

```
function EquilibriumList(const F1,F2 : KSpline ;              // nullclines
                        const Tmin,Tmax : float;             // interval
                        const Step: float):List_of_KPoint;    // discretization step
   var T, Dist1, Dist2 : float              // vectors
       N : integer;
       InternalList : List_of_KPoint;       // dynamic list with closed intervals
   begin
     T:=Tmin;                               // first interval
     Dist1:=F2(T) - F1(T);                  // vector between functions
     repeat  // searching intervals with equilibrium points
         T:=T+Step;                         // next interval
         Dist2:=F2(T) - F1(T);              // vector in the next interval
         If (sign(Dist1) <> sign(Dist2))    // compare signs
            then InternalList.Add(Dist1,Dist2);  // add interval in a list
         Dist1:=Dist2;                      // initialization
     until (T>=Tmax);                       // end of the cycle
     for N:=0 to InternalList.Count-1 do    // cycle across the intervals
       EquilibriumList.Add(Bisection(InternalList[N];  // and calculation the
                                                       // intersections of nullclines
   end; // EquilibriumList
```

Fig.1 Algorithm for calculation of equilibrium points

3.5. Calculation of the eigenvalues of the state matrix.

They are obtained by solving the system:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \qquad (6)$$

The solutions are:

$$\lambda_{1,2} = \frac{a+d}{2} \pm \frac{\sqrt{(a-d)^2 + 4.b.c}}{2} = r \pm i.m \qquad (7)$$

The determination of the type of the points according to their eigenvalues is done respectively to Table 1.

Table T1

case	conditions	type of fixed point
0	*($\lambda 1 < 0$) and ($\lambda 2 < 0$) and (m=0)*	*stable node*
1	*(($\lambda 1 > 0$) or ($\lambda 2 > 0$)) and (m=0)*	*unstable node*
2	*($\lambda 1 * \lambda 2 < 0$) and (m=0)*	*saddle unstable node*
3	*($r < 0$) and (m<>0)*	*stable focus*
4	*($r > 0$) and (m<>0)*	*unstable focus*
5	*($r = 0$) and (m<>0)*	*center point*

3.6. Visualization the "graphs of interest".

Using the interactive features available in Korelia, graphs for visualization are chosen.
   ✓ Graphs of experimental data and identified process;
   ✓ Nullclines of the process;
   ✓ Trajectories of the processes;
   ✓ Phase portrait of the process.
Korelia offers an opportunity to evaluate and analyze the resulting graphs by a number of dynamic parameters [15,16,17].

4. EXAMPLE

An object of analysis is a change in plasma renin activity (PRA) after nicardipine treatment [18,19] of experimental animals. The mathematical model of the process in function of drug dose is presented in [20]. The dose-dependent state-space model for the kinetics of regulation in matrix form is:

$$\left\|\begin{matrix} \dfrac{dy_1(t,d)}{dt} \\ \dfrac{dy_2(t,d)}{dt} \end{matrix}\right\| = \left\|\begin{matrix} 0 & 1 \\ -\omega_0^2(d) & -2\zeta(d)\omega_0(d) \end{matrix}\right\| \cdot \left\|\begin{matrix} y_1(t,d)-y_1^0 \\ y_2(t,d)-y_2^0 \end{matrix}\right\| + \left\|\begin{matrix} 0 \\ K_u(d)\omega_0^2(d) \end{matrix}\right\| \cdot U(d) \qquad (8)$$

where: $y_1(t,d), y_2(t,d)$ -state of the system;

d - applied drug dose [mg/kg b.w.]

y_1^0, y_2^0 - initial conditions

U(d) – the input signal,the applied dose of nicardipine;

$\zeta(d)$ – the damping ratio.

$\omega_0(d)$ - the natural frequency of the system.

$K_u(d)$ – proportionality coefficient.

For dose of 10mg/kg body weight (b.w.) the next parameter values using Korelia were identified:

Initial conditions: $y_1^0 = 7.58$ Parameters: $\zeta(d) = 0.67$

$y_2^0 = 0$ $K_u(d) = 30$

$\omega_0(d) = 0.64$

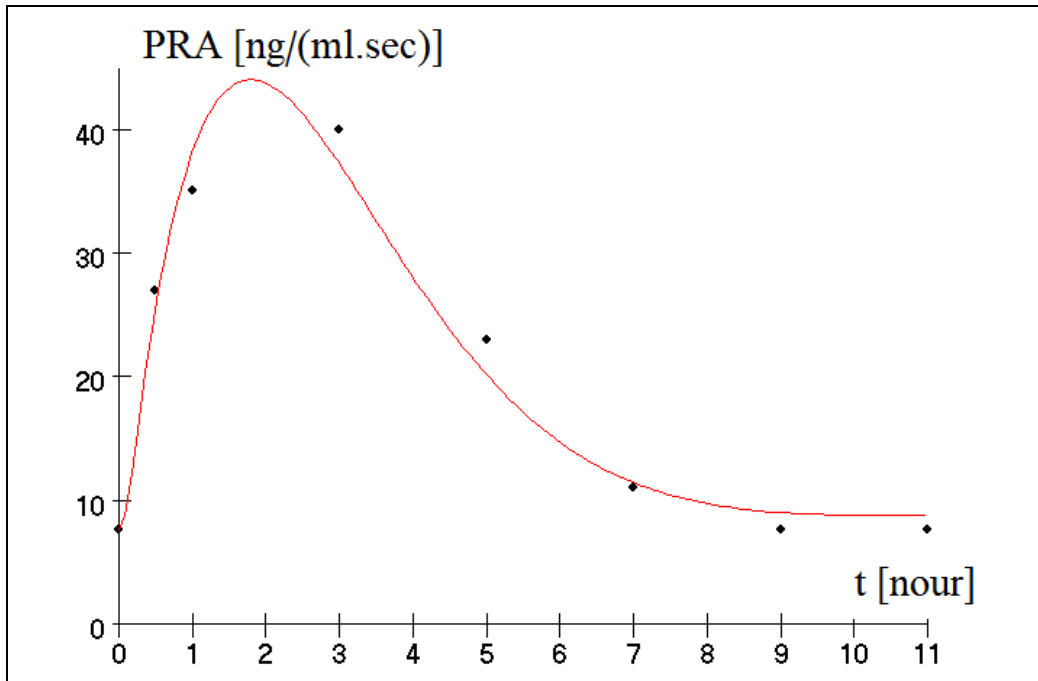The graph of the model is in figure 2.

Fig. 2. Model of PRA after treatment with dose 10 mg/kg b.w.

The independent time parameter t changes from 0 to 11 hours, with 110 sampling points. According to the proposed algorithm, the analysis proceeds in the following order.

1. The differential model described by Korelia-language is on figure 3:

```
Independent t;                          //  natural frequency
w0=0.64                                 // damping ratio
z=0.67
// state space model                    // initial conditions
y01=7.58; y02=0;
y1'=y2;                                 // state space model
y2'=-w0^2*(y1-y01)-2*z*(y2-y02);
```

Fig. 3. Description of the PRA-model

The state matrix is:

$$A = \begin{Vmatrix} 0 & 1 \\ -0.4096 & -0.8576 \end{Vmatrix} \qquad (9)$$

2. The phase-plane is determined by state-space variables. The abscissa is *y1*, and the ordinate is *y2*.

3. The nullclines are defined by the right-side expressions of the state-space. They are calculated in time-interval [0..11]. The nullcline-graphs are on figure 4. In this case, nullcline *y1* matches the abscissa axis.
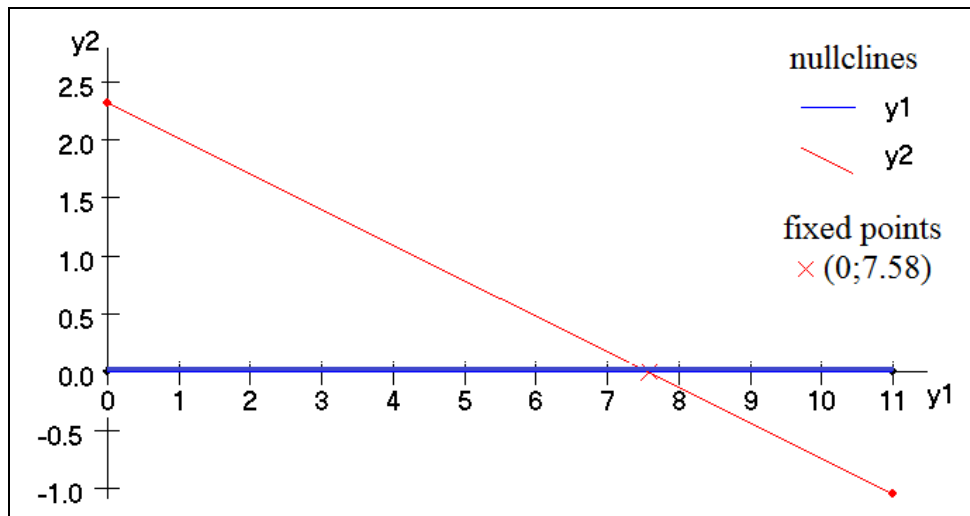

Fig. 4. Nullclines intersects in equilibrium point (0,7.58)

4. Coordinates of the equilibrium point calculated with the algorithm of figure 1 are (0, 7.58). That is the initial condition of the model.

5. Eigenvalues according to Eq.(7) are:

$$\begin{vmatrix} \lambda_1 = -0{,}429 + i0{,}675; \\ \lambda_2 = -0{,}429 - i0{,}675 \end{vmatrix}$$

According to Table 1, they correspond to case 3 and the equilibrium point is a stable focus.

6. Visualization of the graphs of the studied dependencies.

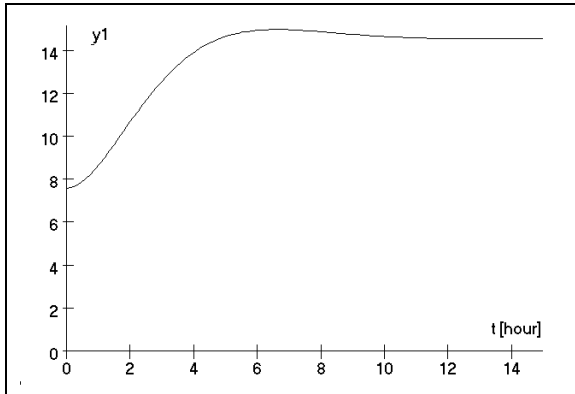The solutions are depicted in figures 5 and 6.

Fig. 5. Trajectory of y₁(t)
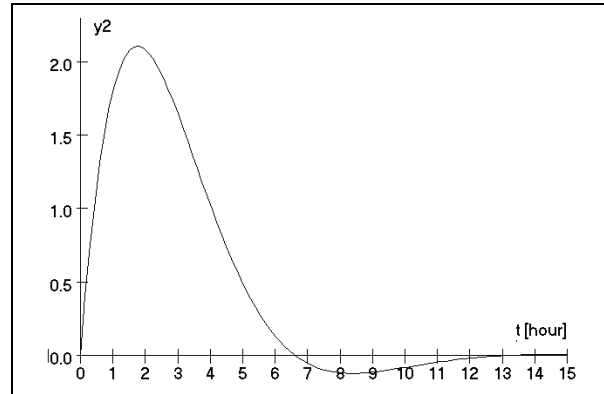

Fig. 6. Trajectory of y₂(t)

The phase portrait of the system is in figure 7. All graphs shown that the process of the PRA activity is stable, when a dose 10mg/kg b.w. is applied.
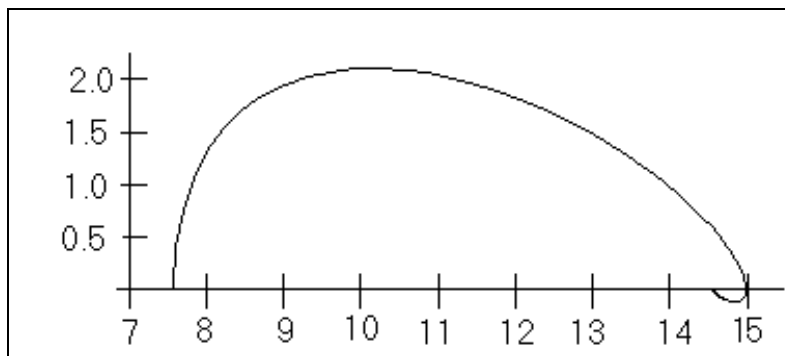

Fig. 7. Phase portrait of the system

5. CONCLUSION

In this work, the possibilities of the KORELIA program for analysis of dynamic processes by the phase plane method are presented. The phase plane method allows analyzing graphically a dynamic process, to find equilibrium points of the system, and to examine system stability at these points without having to solve the differential equation describing the behavior of the system. In this manner the complexity of system analysis is reduced. The dynamics model is described using a specialized programming language. An algorithm for constructing nullclines of the process and for calculating equilibrium points is presented. By determining the vector between nullclines of the process, the intervals in which the particular points are located are found. The exact values are obtained with a bisection algorithm. The type of points is specified by the matrix eigenvectors. All derived dependencies and parameters are displayed in graphical form. The approach is illustrated by a model of plasma renin activity after treatment of experimental animals with nicardipine. The equilibrium point is the initial condition and it is a stable focus.

The proposed in the work approach for the application of phase plane method using Korelia software is easy for implementation. Thanks to friendly-oriented

interface the method can be used by biologists in studying processes from their subject area. On fig.8 a desktop of Korelia with some graphs is shown.
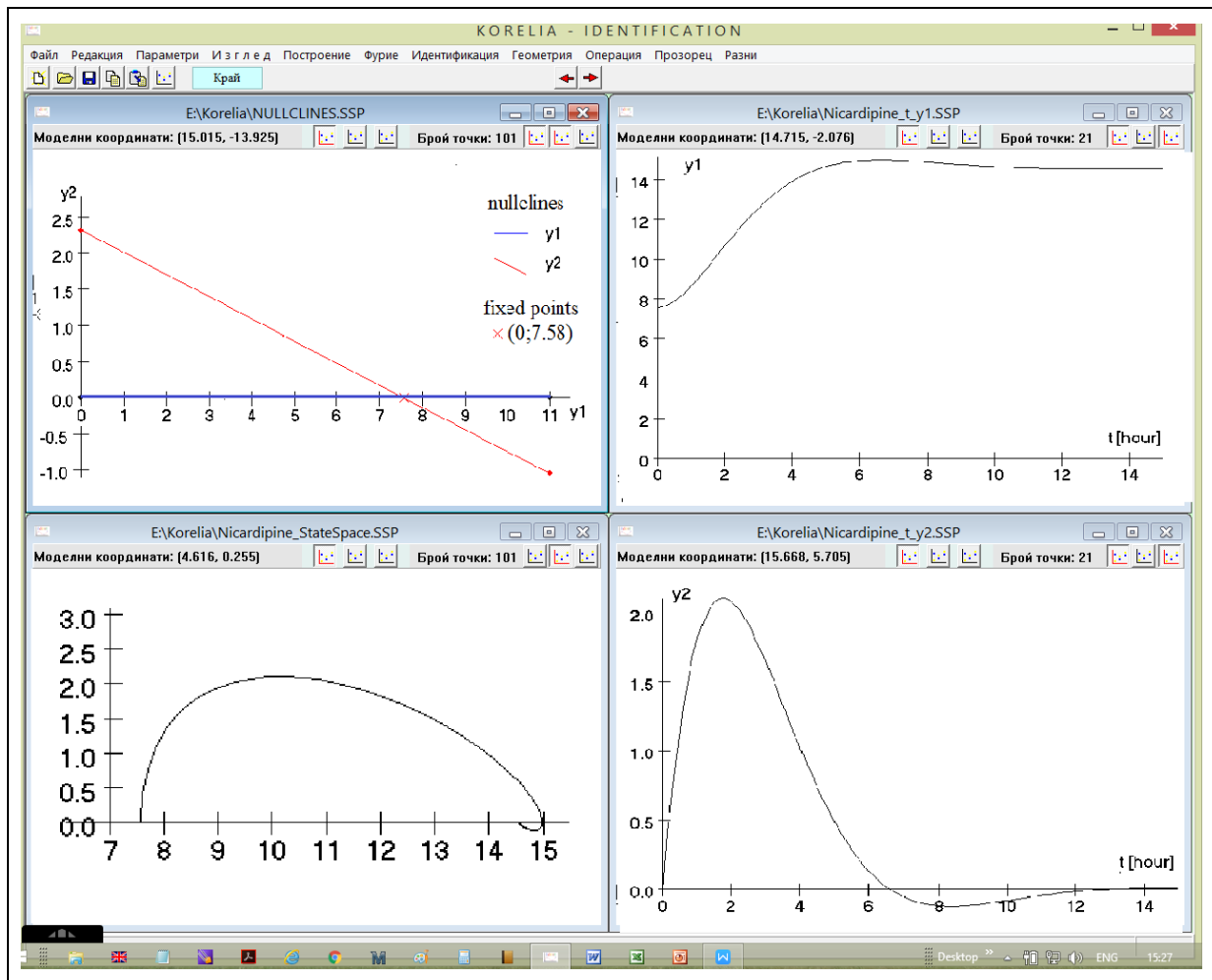

Fig. 8. Korelia desktop and phase plane diagrams

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