Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

Borislav Bedzhev* and Dilyan Dimitrov**

*NMU “V. Levski”, Faculty of Artillery, Air-Defense and CIS, 9700 Shumen, Bulgaria, e-mail: bedzhev@abv.bg

** NMU “V. Levski”, Faculty of Artillery, Air-Defense and CIS, 9700 Shumen, Bulgaria, e-mail: dilyaniv@abv.bg

Introduction

In the paper four methods for synthesis of the so-called generalized skew-symmetric sequences (SSSs) with even lengths, which are phase manipulated (PM) signals with optimal autocorrelation properties, are substantiated. It should be pointed out that up to now classic SSSs with even lengths are no known.

The developed methods could be useful for ensuring the electromagnetic compatibility and cyber security of the sensor networks, exploited today widely in all spheres of industry and transportation.

Necessary Conditions for Existence of Skew-Symmetric Sequences with Even Lengths

One of the infinite PM signal classes, which could be used for reducing of self-interferences (SIs) and multi-access interferences (MAIs), as well as for ensuring a high level of cyber security, comprises the SSSs, invented by M. Golay. Reasons, substantiating this fact, can be summarized as follows. First, the ACF of every SSS has the form:

$$P_{\mu\mu}(r) = \begin{cases} N, & r = 0, \\ \pm U_{sl}(r), N - \lceil r \rceil = odd, \\ 0, N - \lceil r \rceil = even. \end{cases} \quad (1)$$

Necessary Conditions for Existence of Skew-Symmetric Sequences with Even Lengths

In (1) $P_{\mu\mu}(r), -(N-1) \leq r \leq N-1$, is the r-th lobe of the autocorrelation function (ACF) of the SSS (PM signal):

$$\{\mu(k)\}_{k=0}^{N-1} = \{\mu(0), \mu(1), \dots, \mu(N-1)\}. \qquad (2)$$

Second, the length N of the SSSs can be arbitrarily large. As a result, the signal energy, providing the necessary range of radio communication, can be spread in a large spectral band, which minimizes significantly the MAIs.

Third, the restrictions on the samples of $\mu(k), k = 0, 1, \dots, N-1$ of SSSs are not very strong, which makes possible generation of very many different families of SSSs.

Necessary Conditions for Existence of Skew-Symmetric Sequences with Even Lengths

The positive features of the SSSs inspire their intensive exploration during the last fifty years. Despite of all taken efforts at the moment no methods for synthesis of SSSs with even lengths are known. Due to this reason in the last part of this section the necessary conditions for existence of SSSs with even lengths will be investigated in more detail. More specifically, in the case $N \equiv 0 mod 2$, the SSS (2) can be presented as a sum of the following signals, comprising only the samples at even and odd positions respectively:

$$\begin{aligned} \{\mu_e(k)\}_{k=0}^{N-1} &= \{\mu(0), 0, \mu(2), 0, \ \dots, 0, \mu(N-2), 0\}, \\ \{\mu_o(k)\}_{k=0}^{N-1} &= \{0, \mu(1), 0, \mu(3), 0, \ \dots, 0, \mu(N-1)\}. \end{aligned} \quad (3)$$

Necessary Conditions for Existence of Skew-Symmetric Sequences with Even Lengths

The generating functions or Hall polynomials, associated with the SSS (2) and with its reversed and conjugated copy, are:

$$F_{\mu}(x) = F_{e\mu}(x) + F_{o\mu}(x),$$

$$F_{e\mu}(x) = \sum_{k=0}^{N-1} \mu(2k)x^{2k} = \sum_{k=0}^{N-1} \mu_e(k)x^k, \qquad (4), (5)$$

$$F_{o\mu}(x) = \sum_{k=1}^{N} \mu(2k-1)x^{2k-1} = \sum_{k=0}^{N-1} \mu_o(k)x^k.$$

$$x^{N-1}F_{\mu}^{*}(x^{-1}) = x^{N-1}F_{e\mu}^{*}(x^{-1}) + x^{N-1}F_{o\mu}^{*}(x^{-1}),$$

$$F_{e\mu}^{*}(x^{-1}) = \sum_{k=0}^{N-1} \mu_e^{*}(k)x^{-k} = \sum_{k=0}^{(N-2)/2} \mu^{*}(2k)x^{-2k},$$

$$F_{o\mu}^{*}(x^{-1}) = \sum_{k=1}^{N} \mu_o^{*}(k)x^{-k} = \sum_{k=0}^{(N-2)/2} \mu^{*}(2k+1)x^{-2k-1}.$$

In (5) symbol "$*$" stands for complex conjugation.

Necessary Conditions for Existence of Skew-Symmetric Sequences with Even Lengths

The generating functions (Hall polynomials) (4) and (5) allow (1) to be presented by the following polynomial equality (6):

$$\sum_{r=-N+1}^{N-1} P_{\mu\mu}(r)x^r = \left[F_{e\mu}(x) + F_{o\mu}(x)\right]\left[F_{e\mu}^*(x^{-1}) + F_{o\mu}^*(x^{-1})\right] =$$

$$= F_{e\mu}(x)F_{e\mu}^*(x^{-1}) + F_{e\mu}(x)F_{o\mu}^*(x^{-1}) + F_{o\mu}(x)F_{e\mu}^*(x^{-1}) + F_{o\mu}(x)F_{o\mu}^*(x^{-1}).$$

It is not hard to be observed that in (6) the polynomials:

$$\begin{aligned} &F_{e\mu}(x)F_{e\mu}^*(x^{-1}) + F_{o\mu}(x)F_{o\mu}^*(x^{-1}), \\ &F_{e\mu}(x)F_{o\mu}^*(x^{-1}) + F_{o\mu}(x)F_{e\mu}^*(x^{-1}), \end{aligned} \qquad (7)$$

are the generating functions (Hall polynomials) of the “even” and the “odd” parts of the ACF (1) respectively.

Necessary Conditions for Existence of Skew-Symmetric Sequences with Even Lengths

According to classic definition of the SSSs, the “odd” parts of their ACFs must be zero, i.e.:

$$F_{e\mu}(x)F_{o\mu}^{*}(x^{-1}) + F_{o\mu}(x)F_{e\mu}^{*}(x^{-1}) = 0 \qquad (8)$$

must be fulfilled.

Unfortunately, when the length N of the SSS (PM signal) (2) is an even integer, the necessary and sufficient condition (8) never can be satisfied, as the monomials with minimal and maximal degrees in (8)

$$\mu(0)\mu^{*}(N-1)x^{-N+1}, \mu^{*}(0)\mu(N-1)x^{N-1} \text{ (9)}$$

are non-zero.

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

As the ACF's sidelobes $P_{\mu\mu}(-N+1), P_{\mu\mu}(N-1)$ never can be zero, the condition (8) can be fulfilled only approximately when $N \equiv 0\ mod 2$. Obviously, the best approximation of the condition (8), which can be fulfilled by PM signals with even lengths and $|\mu(0)| = |\mu(N-1)| = 1$, is:

$$F_{e\mu}(x)F_{o\mu}^*(x^{-1}) + F_{o\mu}(x)F_{e\mu}^*(x^{-1}) = x^{-N+1} + \\ +0.x^{-N+2} + \cdots + 0.x + \cdots + 0.x^{N-2} + x^{N-1}. \qquad (10)$$

In the sequel four methods for synthesis of such generalized skew-symmetric sequences (GSSSs), satisfying the condition (10), will be substantiated.

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

The Method 1 is based on the following polynomial equality:

$$y^{2L+1}+1=(y+1)(y^{2L}-y^{2L-1}+\cdots-y+1)= \\ =(y+1)\left[\prod_{l=1}^{L}\left(y-w^{2l-1}\right)\right]\left[\prod_{l=1}^{L}\left(y-w^{-2l+1}\right)\right], \quad (11)$$

where w is an arbitrary primitive $(4L+2)$-th root of the unity:

$$w=e^{i\frac{2\pi}{2(2L+1)}n}=e^{i\frac{\pi}{2L+1}n}, i=\sqrt{-1}, (4L+2,n)=1. \quad (12)$$

In (12) the symbol “$(4L+2,n)$” denotes the greatest common divisor of the integers $4L+2$ and n.

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

As $2L$ is an even integer, the zeros $\{w, w^3, \dots, w^{2L-1}, w^{-1}, w^{-3}, \dots, w^{-2L+1}\}$ of the polynomial

$$y^{2L} - y^{2L-1} + y^{2L-2} - \cdots - y + 1 \qquad (13)$$

can be separated in L groups: $\{w,\ w^{-1} = w^*\}, \{w^3,\ w^{-3} = (w^3)^*\}, \dots, \{w^{2L-1},\ w^{-2L+1} = (w^{2L-1})^*\}$ (14). Using (14) and the substitution $y = x^2$, (11) can be transformed as follows:

$$\left[\prod_{l=1}^{L}(y - w^{2l-1})\right]\left[\prod_{l=1}^{L}(y - w^{-2l+1})\right] =$$

$$\left[\prod_{l=1}^{L}(x^2 - v(l))\right]\left[\prod_{l=1}^{L}(x^2 - v^*(l))\right] = x^{4L} - x^{4L-2} + \cdots - x^2 + 1.$$

Here $v(l), l = 1, 2, \dots, L$ is an arbitrary element of the l-th group in (14).

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

On the base of transformation (15) the following proposition is proved.

Proposition 1: Let the samples of the PM signal (2), which length is the even integer $N = 2L + 2$, be defined by the following polynomial equations: $F_{e\mu}(x) = \mu(0) + \mu(2)x^2 + \cdots + \mu(L)x^{2L} = \prod_{l=1}^{L}\left(x^2 - v(l)\right)$, (16)

$F_{o\mu}(x) = \mu(L)x + \mu(L-2)x^3 + \cdots + \mu(0)x^{2L+1} = x^{2L+1}F_{e\mu}(x^{-1})$ (17).

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

Then the sidelobes of its ACF:

1) ever are 0, when $|r|$ is an odd integer, smaller than $N-1$, i.e.

$$P_{\mu\mu}(r) = 0, r = \pm 1, \pm 3, \dots, \pm(N-3); \quad (18)$$

2) are defined by

$$P_{\mu\mu}(r) = P_{\mu_e\mu_e}(r) + P_{\mu_o\mu_o}(r), \\ r = 0, \pm 2, \pm 4, \dots, \pm(N-2), \quad (19)$$

ever, when $|r|$ is an even integer, and

$$P_{\mu\mu}(-N+1) = P_{\mu\mu}(N-1) = 1, \quad (20)$$

when $r = \pm(N-1)$.

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

The Method 2 is based on the following polynomial equality:

$$y^{2L+1} - 1 = (y-1)(y^{2L} + y^{2L-1} + \cdots + y + 1) = \\ = (y-1)\left[\prod_{l=1}^{L}(y - w^{l})\right]\left[\prod_{l=1}^{L}(y - w^{-l})\right], \quad (24)$$

where w is an arbitrary primitive $(2L+1)$-th root of the unity:

$$w = e^{i\frac{2\pi}{2L+1}n}, \quad (2L+1, n) = 1. \quad (25)$$

The zeros $\{w, w^2, \dots, w^L, w^{-1}, w^{-2}, \dots, w^{-L}\}$ of the polynomial $y^{2L} + y^{2L-1} + y^{2L-2} + \cdots + y + 1$ (26), can be separated in L groups: $\{w,\ w^{-1} = w^*\}, \{w^2,\ w^{-2} = (w^2)^*\}, \dots, \{w^L,\ w^{-L} = (w^L)^*\}$ (27).

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

Using (27) and the substitution $y = x^2$, (24) can be transformed as follows:

$$\left[\prod_{l=1}^{L}\left(y - w^{l}\right)\right]\left[\prod_{l=1}^{L}\left(y - w^{-l}\right)\right] =$$
$$= \left[\prod_{l=1}^{L}\left(x^2 - v(l)\right)\right]\left[\prod_{l=1}^{L}\left(x^2 - v^*(l)\right)\right] = \qquad (28)$$
$$= x^{4L} + x^{4L-2} + x^{2L-4} + \cdots + x^2 + 1.$$

In (28) $v(l), l = 1, 2, \dots, L$ is an arbitrary element of the l-th group in (27).

Analogously to the Method 1, the transformation (28) allows the following proposition to be proved.

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

Proposition 2: Let the samples of the PM signal (2), which length is the even integer $N = 2L + 2$, be defined by the following polynomial equations:

$$F_{e\mu}(x) = \mu(0) + \mu(2)x^2 + \cdots + \mu(L)x^{2L} = \\ = \prod_{l=1}^{L}(x^2 - v(l)), \tag{29}$$

$$F_{o\mu}(x) = ix\tilde{F}_{o\mu}(x) = i[\mu(L)x + \mu(L-2)x^3 + \\ + \cdots + \mu(0)x^{2L+1}] = ix^{2L+1}F_{e\mu}(x^{-1}). \tag{30}$$

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

Then the sidelobes of its ACF:

1) ever are 0, when $|r|$ is an odd integer, smaller than $N-1$, i.e.

$$P_{\mu\mu}(r) = 0, r = \pm 1, \pm 3, \dots, \pm(N-3); \qquad (18)$$

2) are defined by

$$P_{\mu\mu}(r) = P_{\mu_e\mu_e}(r) + P_{\mu_o\mu_o}(r),$$
$$r = 0, \pm 2, \pm 4, \dots, \pm(N-2), \qquad (19)$$

ever, when $|r|$ is an even integer, and

$$P_{\mu\mu}(-N+1) = -i, P_{\mu\mu}(N-1) = i, \qquad (20)$$

when $r = \pm(N-1)$.

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

The PM signals, synthesized by the above substantiated methods 1 and 2, have ACFs, possessing only two non-zero sidelodes ($P_{\mu\mu}(-N+1) = P_{\mu\mu}(N-1) = 1$ or $P_{\mu\mu}(-N+1) = -i, P_{\mu\mu}(N-1) = i$) at odd positions (i.e. at positions $r = \pm 1, \pm 3, \ldots, \pm(N-1)$). Consequently, these PM signals could be viewed as GSSSs of the first type. Due to this fact, the methods 3 and 4, which will be substantiated in sequel, can be classified as GSSSs of the second type, as their ACFs possess only four non-zero sidelodes ($P_{\mu\mu}(-N+1), P_{\mu\mu}(-N+3), P_{\mu\mu}(N-3), P_{\mu\mu}(N-1)$) at odd positions.

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

The Method 3 is based on the following polynomial equality:

$$y^{2L} + 1 = \left[\prod_{l=1}^{L}\left(y - w^{2l-1}\right)\right]\left[\prod_{l=1}^{L}\left(y - w^{-2l+1}\right)\right], \quad (34)$$

where w is an arbitrary primitive $(4L)$-th root of the unity:

$$w = e^{i\frac{2\pi}{2(2L)}n} = e^{i\frac{\pi}{2L}n}, \quad (4L, n) = 1. \quad (35)$$

Analogously to (14), the zeros of the polynomial

$$y^{2L} + 1 \quad (36)$$

can be separated in L groups

$$\{w,\ w^{-1} = w^{*}\}, \{w^{3},\ w^{-3} = (w^{3})^{*}\}, \dots, \{w^{2L-1},\ w^{-2L+1} = (w^{2L-1})^{*}\}. \quad (37)$$

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

Using (37) and the substitution $y = x^2$, (34) can be transformed as follows:

$$\left[\prod_{l=1}^{L}\left(y - w^{2l-1}\right)\right]\left[\prod_{l=1}^{L}\left(y - w^{-2l+1}\right)\right] = = \left[\prod_{l=1}^{L}\left(x^2 - v(l)\right)\right]\left[\prod_{l=1}^{L}\left(x^2 - v^*(l)\right)\right] = x^{4L} + 1. \quad (38)$$

In (38) $v(l), l = 1, 2, \dots, L$ is an arbitrary element of the l-th group in (37).

Analogously to the Method 1, the transformation (38) allows GSSSs of the second type with four non-zero ACF sidelodes at odd positions ($P_{\mu\mu}(-N+1) = P_{\mu\mu}(N-1) = P_{\mu\mu}(-N+3) = P_{\mu\mu}(N-3) = 1$) to be synthesized.

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

The Method 4 is based on the following polynomial equality:

$$y^{2L} - 1 = (y-1)(y+1)\left[\prod_{l=1}^{L-1}\left(y - w^{l}\right)\right]\left[\prod_{l=1}^{L-1}\left(y - w^{-l}\right)\right], \qquad (39)$$

where w is an arbitrary primitive $(2L)$-th root of the unity:

$$w = e^{i\frac{2\pi}{(2L)}n} = e^{i\frac{\pi}{L}n}, \quad (2L, n) = 1. \qquad (40)$$

Analogously to (27), the zeros of the polynomial $y^{2L} - 1$ (41)

can be separated in L groups

$$\{w,\ w^{-1} = w^{*}\}, \{w^{2},\ w^{-2} = (w^{2})^{*}\}, \ldots,$$

$$\{w^{L-1},\ w^{-L+1} = (w^{L-1})^{*}\}, \qquad (42)$$

$$\{w^{2L} = w^{0} = 1,\ w^{-L} = w^{L} = -1\}.$$

A Systematization of Methods for Synthesis of Generalized Skew-Symmetric Sequences with Even Lengths

Using (42) and the substitution $y = x^2$, (40) can be transformed as follows:

$$\left[\prod_{l=1}^{L}\left(y - w^{l}\right)\right]\left[\prod_{l=1}^{L}\left(y - w^{-l}\right)\right] = \left[\left((x^2 - v(L))\right)\prod_{l=1}^{L-1}(x^2 - v(l))\right] \times$$

$$\times \left[(x^2 + v(L))\prod_{l=1}^{L-1}(x^2 - v^{*}(l))\right] = x^{4L} - 1.$$

Here $v(l), l = 1, 2, \dots, L$ is an arbitrary element of the l-th group in (42). Analogously to the Method 1, the transformation (43) allows GSSSs of the second type with four non-zero ACF sidelodes at odd positions ($P_{\mu\mu}(-N+1) = P_{\mu\mu}(N-1) = 1, P_{\mu\mu}(-N+3) = P_{\mu\mu}(N-3) = -1$) to be synthesized.

Conclusion

This paper completes the exploration on generalizations of skew-symmetric sequences, began in our previous paper [9]. As a result, four new methods for synthesis of generalized skew-symmetric sequences with even lengths are developed.

The developed methods could be useful for synthesis of PM signals, ensuring the electromagnetic compatibility and cyber security of the smart electronic devices, radio-communication systems and the sensor networks, which have to provide precise geo-spatial information about the moving objects in all branches of transportation and industry.